Optimal self-avoiding paths in dilute random medium

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The combined effects of bond-energy disorder and random-bond exclusion on optimal undirected self-avoiding paths are studied by an original finite-size scaling method in two dimensions. For concentrations of accessible bonds between the undirected and directed percolation thresholds, overhangs do not seem to change the standard self-affine scaling regime characteristic of directed paths. At the undirected threshold the path becomes fractal, with a fractal dimension equal to that of the minimal length path on the infinite cluster backbone. At this point the optimal energy variance scales with time t as t^{ω_c} ($\omega_c = 1.02 \pm 0.05$). Furthermore, ω_c turns out to be exclusively determined by fluctuations in backbone geometry and *not* by disorder in bond energies. This scenario is qualitatively confirmed and extended by renormalization-group calculations on hierarchical lattices. [S1063-651X(97)06102-3]

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I. INTRODUCTION

In the past decade, the study of directed paths (or polymers) in random media (DPRM) has attracted much interest since DPRM are relevant for issues such as domain walls in ferromagnets with random interactions [1], interface growth [2], and the behavior of flux lines in high-T_c superconductors [3,4]. For example, consider an interface between two oppositely magnetized phases of a two-dimensional random Ising ferromagnet. At zero temperature and in the absence of a magnetic field, this interface is a path on the dual lattice, which minimizes the sum of nearest neighbor exchange energies of the bonds it crosses. (For convenience, in the following, we will associate this exchange energy directly with the dual lattice bonds.) We will refer to this as the optimal (energy) path. So far, in most cases, the properties of this interface have been usually studied under the additional restriction of allowing only directed paths [4]. Thus one neglects the possibility of overhangs. However, such a restriction is not expected to be a necessary condition in order to obtain a self-affine geometry at large length scales. For DPRM both the transverse width and the energy fluctuations scale as a power of the longitudinal distance covered by the path. In the case of the transverse width, a power (ζ) less than unity indicates self-affinity [4].

The removal of the directedness constraint in the optimal-path problem can be expected to lead to different physics, especially in situations when disorder is able to induce a crossover from self-affine to fractal geometry. In the magnetic interface case, an example of such a disorder can be found when a fraction $1-p$ of the total number of nearest-neighbor exchange energies is assumed to be infinitely strong. These couplings give rise to bonds, on the dual lattice, which are forbidden for the interface. Besides these forbidden bonds, one still has disorder on the accessible bonds. When the fraction p of accessible bonds is at the percolation threshold value p_c, the optimal path is constrained within the infinite percolation cluster backbone, which is fractal [5]. Moreover, the path cannot be shorter than the minimal length of a path between two points on the percolation cluster. Since these minimal-length paths on a percolation cluster are also fractal, we reach the conclusion that the optimal path must itself be fractal. One can also expect that for $p_c < p < p_{cd}$ (where p_{cd} is the directed percolation threshold) the properties of the optimal path may differ in some respects from those in the DPRM regime. Indeed, in this regime of concentrations each optimal path must necessarily have some overhangs. It is then logical to ask (i) whether this can lead to a new universality class of scaling behavior and (ii) what the precise nature and location of the crossover from the DPRM to the fractal regime is. The possible effect on asymptotic behavior of excluding overhangs is a particularly debated issue in the context of interface growth models [6], which are closely related to DPRM [4].

The present paper is devoted to the investigation of the above issues. Besides studying the possible relevance of configurations with overhangs between the two percolation thresholds, we consider in detail the fractal regime at the undirected percolation threshold and the connection between the scaling properties of the optimal path and the incipient infinite cluster's geometry. We further address the question of whether energy fluctuations of the interface are determined purely by geometry (i.e. by bond exclusion effects) or also by accessible-bond disorder when one is in the fractal regime.

We perform part of our study using a different type of finite size scaling analysis in $d=2$. This is based on transfer-matrix methods for self-avoiding paths [14]. In view of the presence of disorder and of the nonpolynomial complexity of transfer-matrix calculations, the information one can draw from the finite-size analysis is necessarily uncertain and incomplete. To corroborate the conclusions of this analysis and to gain a fuller, though still qualitative, insight, we also perform renormalization-group (RG) calculations on hierarchical models in which we can introduce an appropriate equivalent of paths with overhangs.

This article is organized as follows. In Sec. II we introduce the model, present our finite-size scaling method, and

discuss the results that can be obtained with it. Section III is devoted to a discussion of RG results for a hierarchical version of our model. Finally, in Sec. IV we present our conclusions.

II. OPTIMAL PATHS AND FINITE-SIZE SCALING IN $D=2$

Following the discussion in the introduction, we define the following optimization problem. Consider a two-dimensional square lattice. A fraction $1-p$ of nearest-neighbor bonds is chosen at random and each is made inaccessible (infinite energy). To each remaining bond b, a random (finite) energy E_b is given. These energies are independent, identically distributed random variables taken from a distribution $P(E_b)$. For each given realization of bond energies we can consider the set of (self-avoiding) paths, restricted to have all steps on the accessible bonds, that connect two given points on the same cluster of allowed bonds. To each path W we associate an energy E_W given by:

$$E_W=\sum_{b\in W} E_b, \tag{2.1}$$

where the sum is over the bonds visited by the path. Having in mind a model for magnetic interfaces, we can interpret each energy E_b as $-2J_{b*}$, where J_{b*} is the exchange energy between Ising spins at the ends of the edge $b*$, dual to b. For each such realization of random bond energies, we can determine the path W_0 whose energy is minimal within the set of paths connecting two points on the lattice. We will refer to this path as the optimal path. We are interested in the scaling properties of this path when the distance between the two points becomes large. Our optimization problem makes no sense as soon as p is strictly less than the bond percolation threshold p_c, which, for the present case, equals 1/2. Indeed, the optimal-path properties are defined for us as averages over all bond configurations, subject to the condition that these configurations guarantee a connection with nonzero probability between infinitely distant points. The problem we have defined is a nontrivial generalization of the DPRM problem. We know of only one study in which bond exclusion was considered for the DPRM model; in it the absence of overhangs limited the effects of exclusion and prevented the possibility of crossover to fractal geometry for the optimal path [7].

Indicating by $E_{\min}$ the energy of the optimal path W_0 and by an overbar the average over quenched bond disorder, the scalings of optimal path quantities in terms of longitudinal distance t are

$$\lim_{t\to\infty}\overline{E_{\min}}\sim t^D(1+ct^{\omega_1-1}+\cdots), \tag{2.2}$$

$$\lim_{t\to\infty}\overline{\Delta E}\sim t^{\omega}, \tag{2.3}$$

$$\lim_{t\to\infty}\overline{w}\sim t^{\zeta}, \tag{2.4}$$

$$\lim_{t\to\infty}\overline{N}\sim t^{D}, \tag{2.5}$$

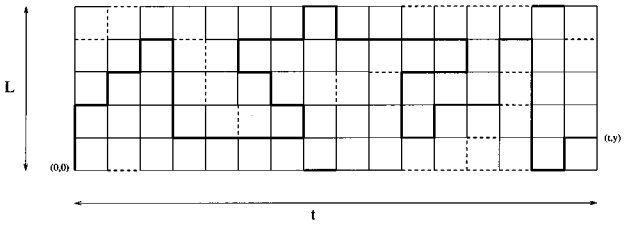

FIG. 1. Strip configuration used in the transfer-matrix calculations for $L=5$. Thick lines represent the path, while dashed lines represent forbidden bonds. Periodic boundary conditions are used.

where ΔE and w are the fluctuations in energy and the transverse width of the optimal path, respectively. These quantities will be defined more precisely below. ω_1 is a subleading exponent and it will be discussed later on in connection with RG calculations on hierarchical lattices. N is the number of steps of the optimal path. Finally, ζ is a roughness exponent, while D is a fractal dimension. In a fractal regime $\zeta=1$ and $D>1$. If the optimal path is self-affine rather than fractal, we expect $\zeta<1$ and $D=1$. Our goal is to determine all the scaling exponents in Eqs. (2.2)–(2.5) as functions of p, in the entire regime $p_c\leqslant p<1$. This program is a real challenge since the presence of overhangs in the possible path configurations makes their number grow very fast with t. Indeed, the counting of undirected paths is a problem of exponential complexity in t. This is in contrast to the polynomial complexity of directed paths.

Let us consider a rectangular strip of length t and width L on our square lattice (cf. Fig 1). Imagine that in each configuration of bonds we sample the energy of the optimal self-avoiding path, if any, contained in the strip and connecting, for example, the lower left corner of the strip with a point (t,y) on the right vertical edge $(0\leqslant y<L)$. We denote by $E(t,L,y)$ the minimal energy for paths reaching the point (t,y) and set

$$E_{\min}(t,L)=\min_y E(t,L,y). \tag{2.6}$$

Energy fluctuations in a given configuration can then be measured [8] by

$$\Delta E(t,L)=\left[\frac{1}{S(t,L)}\sum_y\,[E(t,L,y)-E_{\min}(t,L)]^2\right]^{1/2}, \tag{2.7}$$

where the sum goes over those y values to which at least one connecting path exists [and $S(t,L)$ is the number of these y values in the given disorder configuration].

We denote by $w(t,L)$ the y value for which the minimum in Eq. (2.6) is reached and by $N(t,L)$ the total number of steps in the corresponding path. The quantities $\overline{E_{\min}}$, $\overline{\Delta E}$, $\overline{w}$, and $\overline{N}$ correspond to those in Eqs. (2.2)–(2.5) with the difference that they also depend on L. In a strip of width L the scaling laws (2.2)–(2.5) have to be modified. Since the ratio $\overline{w}/L$ controls the crossover behavior, Eqs. (2.3) and (2.5) become, respectively,

$$\lim_{t\to\infty}\overline{\Delta E(t,L)}\sim t^{\omega}F\left(\frac{t^{\zeta}}{L}\right), \tag{2.8}$$

$$\lim_{t\to\infty}\overline{N(t,L)}\sim t^{D}G\left(\frac{t^{\zeta}}{L}\right). \tag{2.9}$$

In order for these equations to be consistent with Eqs. (2.3) and (2.5), the scaling functions $F(x)$ and $G(x)$ should become constant when x is sufficiently small. On the other hand, at large x one should have $F(x)\sim x^{-\omega/\zeta}$ and $G(x)\sim x^{-(D-1)/\zeta}$, so that for large t the following results hold:

$$\overline{\Delta E(t,L)}\sim L^{\omega/\zeta}, \tag{2.10}$$

$$\overline{N(t,L)}\sim tL^{(D-1)/\zeta}. \tag{2.11}$$

These last two equations can be further understood if one realizes that for $t^{\zeta}\gg L$, the path develops within an essentially one-dimensional lattice. We want to stress the importance of the finite-size scaling relations (2.8) and (2.9) for the present work. In the usual analysis (i.e., for the directed case) the absence of overhangs allows an exact enumeration for large values of t and L and the scaling exponents can be easily obtained directly from Eqs. (2.2)–(2.5). In our numerical work we have used the transfer-matrix approach to self-avoiding paths, introduced by Derrida [14], to find all allowed paths up to a certain t on a given configuration. For each path we calculate the energy [according to Eq. (2.1)] and the number of steps. These results are numerically exact. We then perform an average over randomness on a Monte Carlo basis and from this we calculate quantities such as $\overline{\Delta E(t,L)}$ and $\overline{N(t,L)}$. In this way we are able to get accurate values up to $L=9$. These relatively small values of L make it necessary to introduce the finite-size scaling (FSS) laws (2.10) and (2.11).

Our data were collected for $t\leqslant 100$, and samples of up to 4×10^{6} random bond configurations consistent with Eq. (2.1) were generated to calculate quenched averages. We performed calculations using for $P(E_b)$ both a uniform distribution on $[0,1]$ and a two-valued distribution

$$P(E_b)=q\,\delta(E_b-0)+(1-q)\,\delta(E_b-1). \tag{2.12}$$

The results for exponents did not seem to depend on the distribution taken nor on the values of q in Eq. (2.12), as long as $0<q<1$ strictly. Information on the scaling exponents could be obtained by fitting the data to Eqs. (2.10) and (2.11). In Fig. 2 we show some typical data.

The determination of $(D-1)/\zeta$ was the most successful. For a range of p values including a substantial portion of the interval $[p_c,p_{cd}]$, the extrapolations of this quantity were very close to 0, indicating $D=1$ and thus a self-affine geometry of the path (see Table I). This means that in the region $p_c\lesssim p\leqslant p_{cd}$, overhangs are not able to induce a fractal regime and are not present at sufficiently large length scales. The extrapolations of ω/ζ based on Eq. (2.10) are somewhat less successful. For p values close to 1 ($p\gtrsim 0.8$), ω/ζ extrapolates to 0.40 ± 0.04, which is below the value of the DPRM in two dimensions ($\omega/\zeta=1/2$). Since this underestimation holds starting from $p=1$, e.g. from a case in which DPRM behavior is expected, we interpret it as due to very slow convergence of our data for ω/ζ. However, ω/ζ extrapolates to definitely higher values once $p\sim p_{cd}$ ($\omega/\zeta\approx 0.9$), suggesting that presumably slow crossover to the fractal regime has already started there. It is thus hard to conclude whether the universality class of the self-affine regime in $p_c<p<p_{cd}$ is still the standard DPRM one. However, this seems plausible and further evidence is given in Sec. III.

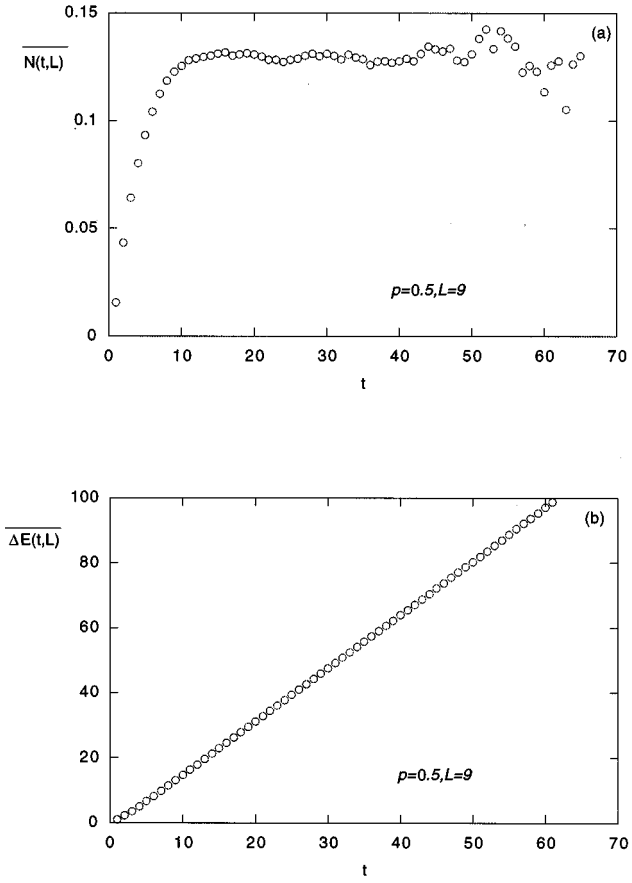

FIG. 2. Numerical results for $L=9, p=p_c$. (a) Results for $\overline{\Delta E(t,L)}$ (b) results for $\overline{N(t,L)}$. Data are for a uniform distribution of bond energies.

A crossover to a fractal regime is detected in both $(D-1)/\zeta$ and ω/ζ when we approach p_c from above. As mentioned already, the crossover for ω/ζ starts much earlier, in fact, causing the poorer analysis for $p\ll p_{cd}$. Exactly at

TABLE I. Results for $\overline{N(t,L)}/t$ versus L and the resulting values for $(D-1)/\zeta$ at $p_c<p=0.6<p_{cd}$ together with similar results for $\overline{\Delta E(t,L)}$ and ω/ζ at $p=0.8$. In both calculations the bond energies were taken from a uniform distribution on $[0,1]$ and represent the result for large t when the asymptotic behavior visible in Fig. 2 has set in.

L	$\overline{N(t,L)}/t$	$(D-1)/\zeta$	$\overline{\Delta E(t,L)}$	ω/ζ
4	1.412		0.3448	
5	1.457	0.141	0.3905	0.56
6	1.488	0.115	0.4277	0.50
7	1.510	0.095	0.4592	0.46
8	1.528	0.089	0.4865	0.43
9	1.538	0.055	0.5113	0.42
$L\to\infty$		-0.01 ± 0.03		0.40 ± 0.04

TABLE II. Data similar to those in Table I, but now at p_c. In this table data are given for a two-valued bond-energy distribution with $q=0.5$. The estimates reported in the text are based also on the comparison of results obtained with different distributions of bond energies.

L	$\overline{N(t,L)}/t$	$(D-1)/\zeta_c$	$\overline{\Delta E(t,L)}$	ω_c/ζ_c
4	1.415		0.8634	
5	1.472	0.177	1.081	1.01
6	1.519	0.172	1.303	1.02
7	1.553	0.143	1.528	1.03
8	1.582	0.139	1.747	1.00
9	1.612	0.159	1.980	1.06
$L\to\infty$		0.11 ± 0.04		1.06 ± 0.04

$p=p_c$ our extrapolations are $(D_c-1)/\zeta=0.13\pm0.04$ and $\omega_c/\zeta_c=1.02\pm0.05$. Assuming $\zeta_c=1$, as appropriate to a fractal regime, these values imply $D_c=1.13\pm0.04$ and $\omega_c=1.02\pm0.05$. These estimates follow from analyzing data for different energy distributions. In Table II some typical data are given for the case of a two-valued energy distribution with $q=0.5$. The determination of D_c is consistent with the existing estimates of the fractal dimension of the shortest path within the backbone of the infinite incipient cluster of percolation [13]. It clearly makes sense that the disorder of accessible bond energies is not able to increase D_c above its lower bound in the fractal regime. However, this disorder could play a substantial role in ω_c. To investigate whether ω_c is determined by the bond exclusion alone or also by the randomness of accessible bonds we made, at p_c, a nonrandom choice for the energies of these bonds. Remarkably enough, we could not detect a significant change in the extrapolated value of ω_c/ζ_c. This suggests that in the fractal regime of the optimal path, geometry (through the effect of bond exclusion) is the only factor determining energy fluctuations. In other words, ω_c comes from the fluctuations in length of the shortest path through a percolation backbone, with negligible contribution from bond-energy fluctuations.

III. RESULTS FOR HIERARCHICAL MODELS

Because of the limitations of our FSS analysis, we also address the main physical using hierarchical models. If not a good quantitative approximation, hierarchical models should at least provide a rather complete qualitative picture of the properties of Euclidean models. Derrida and Griffiths [9] did pioneering work in the application of hierarchical lattices to the study of DPRM [4]. More recently Le Doussal and Machta [10] studied self-avoiding walks in random environment.

Let us consider a hierarchical lattice obtained by following the iteration procedure illustrated in Fig. 3(a): We call the resulting lattice the hierarchical diamond lattice with a diagonal bond (DDHL). So far, only diamond hierarchical lattices without diagonals (DHL) [Fig. 3(b)] have been considered for the study of DPRM [11]. However, since in a DHL all paths joining the extremes have the same length, this lattice does not allow a distinction between ''directed'' and ''undirected'' paths.

In our DDHL we regard as directed a path of minimal

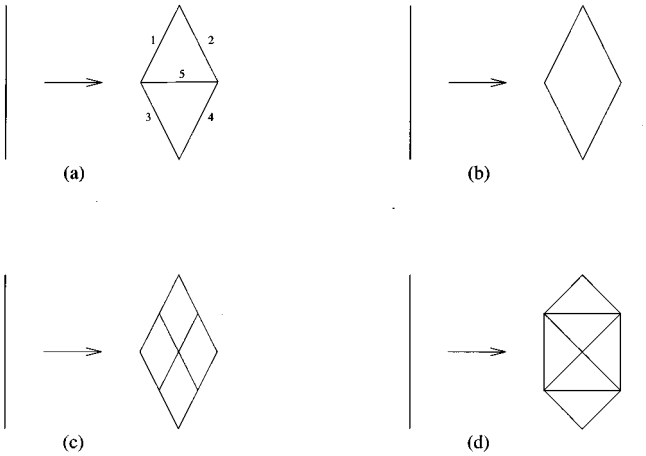

FIG. 3. Iterative construction of the two hierarchical lattices: (a) DDHL lattice (the bonds are numbered and the paths to be considered in the iterative RG calculation are 1,3; 1,5,4; 2,4; 2,5,3); (b) DHL lattice, (c) hierarchical lattice with $p_c=0.708\,30\ldots$ and $p_{cd}=0.725\,78\ldots$, and (d) hierachical lattice with $p_c=0.549\,84\ldots$ and $p_{cd}=0.551\,94\ldots$.

length joining the extrema, i.e., a path never passing through a diagonal at any stage of lattice construction. Paths not satisfying this condition will be called undirected. On the DDHL we can also distinguish between a directed percolation threshold p_{cd} and an undirected one p_c. The directed threshold is reached when the concentration p of accessible bonds ceases to guarantee connectivity of the infinite lattice ends through directed paths. Simple RG calculations give $p_{cd}=(\sqrt{5}-1)/2$ and $p_c=1/2$ for bond percolation on the DDHL.

To set up our RG treatment, we consider the joint energy-length probability distribution for optimal paths at each level n of lattice construction. $n=0$ corresponds to a single bond (end-to-end distance $2^\circ=1$) and, consistently with the discussion in Sec. II, we must choose

$$P_0(E,N)=\delta_{N,1}[(1-p)\,\delta(E-\infty)+pP(E)]. \qquad (3.1)$$

Here $P(E_b)$ is again taken as either a uniform distribution on $[0,1]$ or a two-valued distribution such as that given by Eq. (2.12). In Eq. (3.1) N is the optimal path length. P_{n+1} can be recursively constructed from P_n as

$$P_{n+1}(E,N)=\sum_{\{N_\alpha\}}\int\prod_{\alpha=1}^{5}\left[dE_\alpha P_n(E_\alpha,N_\alpha)\times\delta\left(E-\min_W\sum_{\beta\in W}E_\beta\right)\delta_{N,\sum_{\gamma\in W_{\min}}N_\gamma}\right]. \qquad (3.2)$$

In Eq. (3.2) α labels the five bonds of a diamond cell with diagonal [Fig. 3(a)]. W are all the distinct self-avoiding, undirected paths crossing the cell, and $W_{\min}$ indicates the optimal one among them, whose steps are labeled by γ. Of course, like P_0, P_n contains a term proportional to $\delta(E-\infty)$, corresponding to the events in which there is no path crossing the n-th level lattice. By construction, the co-

efficient of this term coincides with the nth iterated of the RG transformation for undirected percolation [5] on DDHL, in which each $n=1$ diamond is scaled to a single bond:

$$p' = (2p^3 - 5p^2 + 2p + 2)p^2. \tag{3.3}$$

Thus, when $p > p_c$ the concentration of accessible bonds grows to $p=1$ under iteration, while at $p=p_c=1/2$ it remains fixed.

Iteration of Eq. (3.2) cannot be performed exactly. So we used a simple and very accurate Monte Carlo sampling strategy to follow the evolution of P_n.

We start by collecting a large sample of energies and lengths distributed according to Eq. (3.1): $\{(E_i^{(0)}, N_i^{(0)})\}, i=1,2,\ldots,i_{\max}(i_{\max} \leq 10^6)$. From this sample, groups of five pairs are collected at random and each pair is assigned to the αth bond of the diamond cell ($\alpha=1,2,\ldots,5$). For this arrangement the total energy and length of the optimal path in the cell is sought and stored as one of the elements $(E_i^{(1)}, N_i^{(1)})$, building up the sample for P_1. This procedure can be iterated several times before serious problems arise due to the limited accuracy and sampling.

Knowledge of P_n allows the determination of quantities such as $\overline{E_{\min}}$, $\overline{\Delta E}$, or the average optimal path length N as a function of the initial conditions and of $t \equiv 2^n$. We could normally push our calculations to $n=20$. The advantage of the iterative procedure is that it allows the extraction of reliable information on behaviors at very large length scales without actually implementing a simulation of the system of the corresponding size. (This would require at least some 5^{20} random bond energies for $n=20$.) This is the advantage of the RG procedure, which exploits the hierarchical nature of the problem for the iteration of the sampling of P_n.

For our model on DDHL we first considered the case $p > p_{cd}$, with energy disorder specified by $q=1/2$ [see. Eq. (2.12)]. With respect to the DHL case, the asymptotic regime here is not modified by the presence of undirected configurations. Indeed, we find $\omega = \ln(1.230\ldots)/\ln(2) \sim 0.30$, which coincides, within the numerical accuracy, with the value obtained in Ref. [9]. On DHL this ω is expected to correspond to the DPRM value $\omega=1/3$ on a square lattice [4].

By assuming validity of the scaling relation [1,4]

$$\omega = 2\zeta - 1 \tag{3.4}$$

linking energy to length fluctuations, one also gets a roughness exponent $\zeta \sim 0.65$, obviously close to the DPRM value 2/3 [12]. This result for $p > p_{cd}$ is certainly consistent with what is expected to happen on Euclidean lattices, but was never explicitly verified so far in the hierarchical context, to our knowledge. At $p=p_{cd}$ the path on DDHL behaves differently from that on DHL. Indeed, for DHL, $p=p_{cd}$ is the percolation threshold, below which the optimization problem loses meaning. One can verify that, at $p=p_{cd}$, $\omega \sim 0.49 \pm 0.02$ on DHL, while $\omega \sim 0.30$ remains for DDHL. A value of $\omega \sim 0.50$ was already found for $T>0$ on DHL in Ref. [7], where it was shown to be in agreement with the transfer-matrix result, obtained in the same paper, for directed paths on a square lattice ($\omega = 0.50 \pm 0.01$).

Our results for the DDHL show that the directed threshold is not ''felt'' at all by the undirected optimal path. We find that, in the whole region $p_c < p \leq p_{cd}$, ω stays at 0.30. This means that, even if optimal path configurations now contain overhangs, these do not affect the large-scale behavior. Further support of the conclusion that the optimal path on the DDHL is self-affine for $p_c < p \leq 1$ comes from a computation of the fractal dimension D, of the path, based on the scaling $\overline{N} \sim (2^n)^D$. With very high precision we find $D=1$ over the whole range. Notice that on the DHL, which in DDHL is equivalent to requiring directedness, $D=1$ would hold by construction. Thus we conjecture that self-affine DPRM behavior with $\omega=1/3$ and $\zeta=2/3$ should hold for optimal self-avoiding paths on square lattice as long as $p > p_c$.

Of particular interest is the case $p=p_c=1/2$ on the DDHL. For this threshold case we find $\omega_c = 1.04 \pm 0.05$. Assuming validity of Eq. (3.4), $\omega=1$ would correspond to $\zeta=1$, the limiting value at which self-affinity changes into fractality. However, it is not clear whether Eq. (3.4) should hold in this situation. For example at the directed percolation threshold in two dimensions directed paths do not satisfy such a scaling law [7].

The fractal character of the path at $p=p_c$ is demonstrated by our result $D_c = 1.085 \pm 0.005$. In fact, the optimal path at threshold is constrained to develop within the backbone of the infinite incipient percolation cluster, and we can try to identify D_c among the fractal dimensions of the backbone on the DDHL. As in the case of the Euclidean lattice discussed in Sec. II, a natural candidate is the dimension $D_{\min}$ of the minimal length path across the backbone. One way to test this is to compare D_c values for disordered bond energies ($q=1/2$, for example) with a case with no disorder ($q=0$, for example). In the latter case, the optimal path has to coincide with the minimal-length path; thus $D_c = D_{\min}$ must hold. Remarkably, at $p=p_c$ we find no appreciable differences in the D_c and ω_c estimates ($\omega_c = 1.04 \pm 0.05$) for $q=1/2$ and $q=0$, $\omega_c = 1.04 \pm 0.05$. As already found in the Euclidean case, the fact that ω_c also remains unchanged when $q=0$ means that energy fluctuations in this regime are dominated by the geometrical backbone disorder: thus we conjecture that ω_c has the geometrical interpretation of describing the fluctuations of minimal chemical distance within the backbone. The presence or absence of energy disorder in the backbone bonds does not affect the value of ω, which is just a direct manifestation of the incipient infinite cluster fluctuating statistical geometry.

Further evidence that $D_c = D_{\min}$ comes from an exact RG evaluation of $D_{\min}$ for percolation on DDHL. The result is $D_{\min} = \ln(17/8)/\ln 2 = 1.089$. This determination can be done analytically within the RG scheme, by appropriately weighing, at $p=p_c$, the lengths of all possible paths crossing a percolating cell and performing the average.

In order to obtain another example of the predictive value of this kind of model calculations it is worth checking the robustness of ω_c with respect to modifications of the hierarchical lattice. We used other lattices, with more complicated cells, reproducing more features of the square lattice. Two examples are reported in Figs. 3(c) and 3(d). With similar, somewhat more laborious RG calculations, we could confirm the scenario described above for the DDHL. In particular, at

the percolation thresholds we determined $\omega_c = 1.02 \pm 0.05$ [Fig. 3(c)] and $\omega_c = 1.01 \pm 0.05$ [Fig. 3(d)], consistent with the DDHL result. We should expect the results for ω_c in the case of the lattices in Figs. 1(c) and 1(d) to represent a better approximation of the Euclidean exponent.

To elucidate further the meaning of ω in the case of fractal optimal path, we also checked how Eq. (2.2) changes in this case. Our data for $\overline{E_{\min}}$ on DDHL could be fitted very well for $L = 2^N$ by the form

$$\overline{E_{\min}}(L) \sim aL^{D_{\min}} + bL^{\omega_{1c}}, \quad L \gg 1, \tag{3.5}$$

with $D_{\min} = 1.08$ and $\omega_{1c} = 1.01 \pm 0.03$. The agreement of ω_{1c} in Eq. (3.5) with the exponent ω_c describing $\overline{\Delta E}$ suggests an asymptotic distribution

$$Q_n(E) = \sum_N P_n(E,N) = \frac{1}{L^{\omega_c}} f\left(\frac{E - E_{\min}(L)}{L^{\omega_c}} \right) \quad (E < \infty) \tag{3.6}$$

of optimal path energies. Thus ω_c should be the only scaling dimension for energy in the fractal regime at p_c.

IV. CONCLUSION

In this paper we carried out a systematic study of optimal undirected self-avoiding paths in a random medium with a fraction of forbidden bonds. We wanted to establish to what extent removal of directedness and bond exclusion up to the percolation threshold can lead to different physics for the optimal path solution.

In view of the considerable complexity associated with counting undirected paths, we had to devise an original FSS analysis to extract information on the optimal-path scaling in two dimensions. The main advantage of this strategy is that, by confirming the path within strips of width L, useful extrapolations can already be made with relatively small L. We are confident that our FSS approach will be useful for other problems of complex optimization.

Our FSS results on a square lattice suggest that crossover to fractal behavior ($D > 1$) occurs for the path only at $p = p_c$. It is less clear, however, whether the self-affine regime with $p_c \leqslant p \leqslant p_{cd}$, where overhangs are necessarily present, still belongs to the DPRM universality class, specifically whether the exponent ω does not change. Convergence of the ratio ω/ζ from our FSS seems to be much poorer than that of $(D-1)/\zeta$.

The results for the fractal regime at $p = p_c$ are more conclusive. It appears most plausible that D_c coincides with the dimension of the shortest path in the percolation cluster backbone $D_{\min}$ and $\omega_c = 1.02 \pm 0.05$ merely reflects the length fluctuations of this shortest path. Thus, in the fractal optimal-path regime the bond-energy probability distribution P plays no appreciable role in the sense of determining a nontrivial stable law for minimal energy fluctuations. Geometry alone seems to control the optimal-path scaling regime at the percolation threshold, in contrast with what happens in self-affine regimes.

All the above conjectures were qualitatively reinforced and extended by means of our analysis of undirected paths on DDHL and similar lattices. Besides confirming the crucial role of percolative geometry for the scaling of undirected optimal paths at p_c, the hierarchical calculations gave a nice demonstration that nothing dramatic happens at $p = p_{cd}$: the scaling exponent of energy fluctuations remains the same as for the DPRM case over the whole range $p_c < p \leqslant p_{cd}$. Thus we expect that, also in the Euclidean case, overhangs are not important at large scales in the regime $p < p \leqslant p_{cd}$ and do not affect whatsoever the scalings of the optimal path.

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